Eur. Phys. J. B **26**, 191–198 (2002)
DOI: 10.1140/epjb/e20020080

THE EUROPEAN
PHYSICAL JOURNAL B
EDP Sciences

On the steady motions of a flat domain wall in a ferromagnet

P. Podio-Guidugli[a] and G. Tomassetti

Dipartimento di Ingegneria Civile, Università di Roma "Tor Vergata", Via di Tor Vergata, 110 - 00133 Roma, Italy

Received 16 November 2001

Abstract. A new derivation is presented of Walker's exact solution to Gilbert equation, a solution which mimicks the travelling-wave motion of a flat domain wall at 180°. It is shown that a process during which the working of the applied magnetic field exactly compensates dissipation (the Walker condition) exists both under the constitutive circumstances considered in the standard Gilbert equation and when either the internal free-energy or the dissipation, or both, are generalized by the introduction of higher-gradient terms; but that such a process cannot solve the generalized Gilbert equation. It is also shown that, when dry-friction dissipation is considered and a suitable magnetic field is applied, the associated Gilbert equation has a Walker-type solution mimicking a flat wall, at 90° this time, which however does not satisfy the Walker condition.

PACS. 75.40.Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.) – 75.60.Ch Domain walls and domain structure

1 Introduction

Consider an infinite ferromagnetic body partitioned into two domains by a flat wall parallel to the easy axis $\mathbf{e}$, and suppose that the wall be a 180°−wall, *i.e.*, that the magnetization field $\mathbf{m}$, while having constant direction in each domain, rotate from $\mathbf{e}$ to $-\mathbf{e}$ across the wall thickness Δ (Fig. 1). It was Landau and Lifshitz [1], in 1935, who first put together a variational mathematical model for the statics of this idealized physical situation, and gave an explicit analytical solution for the spatial dependence of the magnetization inside the wall, in the absence of an applied magnetic field. Their solution pictures a Bloch wall, in that the magnetization rotates in a plane parallel to the wall; and it effectively concentrates the rotation about $x = 0$ (Fig. 3), allowing for an estimate of Δ, although it actually spreads the rotation over the whole axis perpendicular to the wall. In the same path-breaking paper [1], Landau and Lifshitz also considered the case when an external magnetic field parallel to the easy axis sets the wall in motion; they derived an approximate solution of the traveling-wave form

$$\mathbf{m} = \mathbf{m}(\xi)\,, \quad \xi := x - vt, \tag{1}$$

with which they were able to estimate the dynamical magnetic permeability of a ferromagnet. Some three decades later, in his doctoral thesis, Walker [2,3] furnished an exact solution of the form (1) to the classical (Landau-Lifshitz and) Gilbert [4] evolution equation for the magnetization as adapted to the flat-wall case. Walker's solution, being explicit, permits us to dispose of the somewhat casual asymptotics used by Landau and Lifshitz to justify their approximations. Remarkably, constants apart, Walker's dynamic solution depends on the current variable ξ just as Landau-Lifshitz' static solution depends on the spatial coordinate x (Sect. 4); moreover, during a Walker's evolution, as noted in passing on page 275 of [3], dissipation is exactly compensated by external working. This fact draws attention to the class of the Walker processes, that is to say, those evolution processes of the magnetization field during which such a compensation condition is fulfilled. One may then ask whether, for ferromagnets whose constitutive response is more general than the standard response, it would still be true that, granted the compensation condition, the traveling-wave solutions to the dynamic problem have the same form as the variational solutions to the associated static problem. One may also ask whether there are other Walker processes – in addition to the steady propagation motions of type (1) discussed below – that solve the standard Gilbert equation, or generalizations of it. It was to answer questions of this type that we undertook the work whose first results we here present.

Our paper begins by a quick description of the mathematical model leading to a generalized version of the Gilbert equation (see Sect. 2; details are found in [5]). We derive a scalar consequence of this equation, the Liapounov relation establishing that the dissipation and the time rate of a body's total energy, internal plus external, must sum to null in every possible process; in particular, it follows from the Liapounov relation that the body's internal energy is conserved during a Walker process. Next, after in Section 3 we collect some preparation results, we introduce our procedure to derive the Walker solution to the classical

[a] e-mail: ppg@uniroma2.it

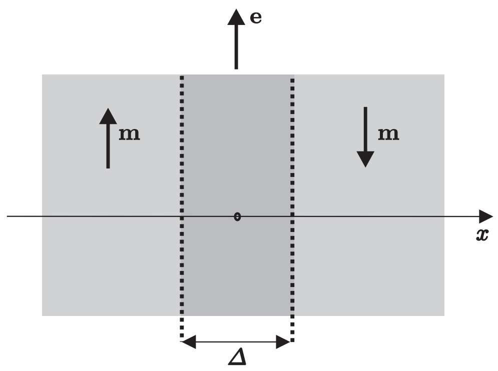

Fig. 1.

Gilbert equation (Sect. 4). By applying this procedure, we show in Section 5 that, when the Gilbert equation is generalized by the addition of terms due to either higher-order exchange energy or exchange dissipation (or both) [5–10], the Walker processes retain their form but do not solve the generalized Gilbert equation. In our final section we apply our procedure again in the case when a dry-friction dissipation term is added to the standard Gilbert equation [11,5]. We show that, if the applied magnetic field has suitable, nonvanishing components in the directions orthogonal to the easy axis, then there are exact solutions of the Gilbert equation picturing 90°−walls; these solutions, however, are not Walker processes.

2 The Gilbert equation

In a saturated, undeformable ferromagnet occupying the region Ω, the evolution of the magnetization vector is ruled by the following equation:

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m} \times (\mathbf{h} + \mathbf{d})\,, \tag{2}$$

where the constant $\gamma < 0$ is the *gyromagnetic ratio*, $\mathbf{m}$ the *magnetization* (a unit vector, at saturation), $\mathbf{h}$ the *magnetic field*, and $\mathbf{d}$ the *dissipation field*[1]. This equation is a formal generalization of the classical model equation due to Landau and Lifshitz and Gilbert, a generalization that can be given a precise physical status [5,7,8]; provided $\dot{\mathbf{m}} \neq \mathbf{0}$, it can be written as the following system of two scalar equations:

$$\begin{aligned} -\gamma^{-1}\dot{\mathbf{m}}\cdot\dot{\mathbf{m}} &= (\mathbf{h}+\mathbf{d})\cdot\mathbf{m}\times\dot{\mathbf{m}}\,,\\ 0 &= (\mathbf{h}+\mathbf{d})\cdot\dot{\mathbf{m}}\,. \end{aligned} \tag{3}$$

The magnetic field $\mathbf{h}$ is the variational derivative of the *free-energy functional* $\Psi\{\mathbf{m}\}$, while the dissipation field $\mathbf{d}$ is the variational derivative of the *dissipation potential* $X\{\dot{\mathbf{m}}\}$:

$$\mathbf{h} = -\,\delta_{\mathbf{m}}\Psi\,,\quad \Psi\{\mathbf{m}\} = \int_\Omega \psi(\mathbf{m},\nabla\mathbf{m},\nabla\nabla\mathbf{m})\,, \tag{4}$$

$$\mathbf{d} = -\,\delta_{\dot{\mathbf{m}}}X\,,\quad X\{\dot{\mathbf{m}}\} = \int_\Omega \chi(\dot{\mathbf{m}},\nabla\dot{\mathbf{m}})\,. \tag{5}$$

[1] The notation we here use is only reminiscent of the notation commonly used in the literature on the physics of magnetized matter; in the Appendix we discuss it in the light of the latter, and pay due attention to the relevant dimensional issues.

Thermodynamic compatibility is guaranteed if the *dissipation density* is non-negative:

$$d := \partial_{\dot{\mathbf{m}}}\chi\cdot\dot{\mathbf{m}} + \partial_{\nabla\dot{\mathbf{m}}}\chi\cdot\nabla\dot{\mathbf{m}} \geq 0\,. \tag{6}$$

Provided that the appropriate homogeneous Neumann conditions prevail at the boundary of Ω, it follows from $(3)_2$, (4), and (5), that

$$\dot{\Psi} + \int_\Omega d = 0\,, \tag{7}$$

a relation that embodies the *Liapounov structure* intrinsic to the Gilbert equation, a structure that all of its generalizations must of course retain. We split the magnetic field and the free energy as follows:

$$\mathbf{h} = \mathbf{h}_{ext} + \mathbf{h}_{int}, \tag{8}$$

$$\psi = \psi_{ext} + \psi_{int},\quad \psi_{ext} = -\,\mathbf{h}_{ext}\cdot\mathbf{m}, \tag{9}$$

$$\Psi = \Psi_{ext} + \Psi_{int},\quad \Psi_{ext}\{\mathbf{m}\} = -\int_\Omega \mathbf{h}_{ext}\cdot\mathbf{m}\,. \tag{10}$$

We regard $\mathbf{h}_{ext}$ as a control field, at our disposal to generate one or another space-time evolution for the magnetic field in a given region Ω. As anticipated in the Introduction, we are especially interested in finding circumstances when *the external working balances the dissipation pointwise*:

$$-\,\mathbf{h}_{ext}\cdot\dot{\mathbf{m}} + d = 0\,.^2 \tag{11}$$

Under such circumstances,

$$\dot{\Psi}_{ext} + \int_\Omega d = 0\,, \tag{12}$$

and hence, due to the Liapounov relation (7), *the internal free-energy is globally conserved*:

$$\dot{\Psi}_{int} = 0\,. \tag{13}$$

More precisely, we are interested in finding solutions, if any, of the generalized Gilbert equation (2) by looking into the set of the *Walker processes*, *i.e.*, the solutions of the scalar equation (11); we refer to the latter as the *Walker condition*. We shall first consider the classical choices of ψ_{int} and χ to which both Landau and Lifshitz and Gilbert confined themselves, then some of the more general choices that have been recently suggested.

3 Preliminaries to the Walker solution

3.1 Standard dissipation and free energy

As to the density of the dissipation potential, the standard choice is

$$\chi = \frac{1}{2}\,\mu\,|\dot{\mathbf{m}}|^2\,,\quad \mu > 0\,, \tag{14}$$

[2] Trivially, this compensation condition implies that, if the external working is null, so is dissipation; but then, as it is not difficult to deduce from (6), thermodynamics only allows for static processes. As a matter of fact, the Landau-Lifshitz solution is a *static* Walker process.

the so-called *relativistic* dissipation. According to (5), the relativistic dissipation field has the form

$$\mathbf{d} = -\mu\,\dot{\mathbf{m}}\,, \tag{15}$$

and hence, by the definition in (6),

$$\mathbf{d}\cdot\dot{\mathbf{m}} + d = 0\,. \tag{16}$$

As to the internal free energy, the standard choice is

$$\psi_{int} = \psi_e + \psi_a + \psi_s\,, \tag{17}$$

where ψ_e, ψ_a, and ψ_s are the *exchange*, *anisotropy*, and *stray* energies; the corresponding splitting of the internal magnetic field is

$$\mathbf{h}_{int} = \mathbf{h}_{ext} + \mathbf{h}_a + \mathbf{h}_s\,. \tag{18}$$

Specifically, the exchange energy is

$$\psi_e = \frac{1}{2}\,\alpha\,|\nabla\mathbf{m}|^2\,, \quad \alpha > 0\,, \tag{19}$$

with

$$\mathbf{h}_e = \alpha\triangle\mathbf{m}\,. \tag{20}$$

The anisotropy energy is

$$\psi_a = -\,\frac{1}{2}\,\beta\,(\mathbf{m}\cdot\mathbf{e})^2\,, \tag{21}$$

where the unit vector $\mathbf{e}$ gives the direction of the *easy axis* ($\beta > 0$) or the orientation of the *easy plane* ($\beta < 0$). Accordingly,

$$\mathbf{h}_a = \beta\,(\mathbf{m}\cdot\mathbf{e})\mathbf{e}\,. \tag{22}$$

Finally, the stray magnetic field is classically taken to be the unique square-integrable solution $\mathbf{h}_s(\mathbf{m})$ of the *quasi-static Maxwell equations*

$$\begin{aligned} \operatorname{rot}\mathbf{h}_s &= 0\,,\\ \operatorname{div}\mathbf{h}_s &= -\operatorname{div}\mathbf{m}\,, \end{aligned} \tag{23}$$

in the whole space. The related energy density (also called the demagnetization energy) is

$$\psi_s = -\,\frac{1}{2}\,\mathbf{h}_s(\mathbf{m})\cdot\mathbf{m}\,. \tag{24}$$

In the next subsection we make the dependence of the stray field $\mathbf{h}_s$ on the magnetization $\mathbf{m}$ explicit, for the one-dimensional problem we here address.

3.2 Stray-field energy in a flat wall

Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ an orthonormal Cartesian frame with $\mathbf{c}_1$ perpendicular to the wall and $\mathbf{c}_3 = \mathbf{e}$. In the case of a flat wall, the magnetization vector depends (beside possibly for the time t, which is kept fixed in our present discussion) on the one spatial coordinate $x \equiv x_1$:

$$\mathbf{m} = \mathbf{m}(x,t)\,. \tag{25}$$

As shown in the Appendix, the associated stray field must be parallel to $\mathbf{c}_1$, and have the form:

$$\mathbf{h}_s = -((\mathbf{m}\cdot\mathbf{c}_1) + c)\mathbf{c}_1\,, \tag{26}$$

where c is spatially constant. If we take $c = 0$,[3] the stray-field energy (24) takes the form:

$$\psi_s = \frac{1}{2}\,(\mathbf{m}\cdot\mathbf{c}_1)^2\,. \tag{27}$$

3.3 Standard dissipation balance and Gilbert equation

With (26), relations (17) and (18) for the density of internal energy can be given the following explicit forms:

$$\psi_{int} = \frac{1}{2}\,\alpha\,|\nabla\mathbf{m}|^2 + \frac{1}{2}\,\mathbf{T}\mathbf{m}\cdot\mathbf{m}\,, \tag{28}$$

with

$$\mathbf{T} := -\beta\,\mathbf{e}\otimes\mathbf{e} + \mathbf{c}_1\otimes\mathbf{c}_1\,, \tag{29}$$

where use has been made also of (20) and (22). Substitution of (20, 22) and (26) into (18) gives:

$$\mathbf{h}_{int} = \alpha\triangle\mathbf{m} + \beta(\mathbf{e}\cdot\mathbf{m})\mathbf{e} - (\mathbf{c}_1\cdot\mathbf{m})\mathbf{c}_1\,, \tag{30}$$

or rather, with the use of (29),

$$\mathbf{h}_{int} = \alpha\triangle\mathbf{m} - \mathbf{T}\mathbf{m}\,. \tag{31}$$

In addition, for the external magnetic field driving the wall motion we choose

$$\mathbf{h}_{ext} = h\,\mathbf{e}\,, \quad h = \text{const.} \tag{32}$$

In conclusion, the Walker equation (11) becomes

$$(-\,h\,\mathbf{e} + \mu\,\dot{\mathbf{m}})\cdot\dot{\mathbf{m}} = 0\,.^4 \tag{33}$$

Moreover, the generalized Gilbert equation (2) reduces to the standard form

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m}\times(\mathbf{h}_{int} + h\,\mathbf{e} - \,\mu\,\dot{\mathbf{m}})\,; \tag{34}$$

the equivalent scalar system (3) is

$$\begin{aligned} -\,\gamma^{-1}\dot{\mathbf{m}}\cdot\dot{\mathbf{m}} &= (\mathbf{h}_{int} + h\,\mathbf{e})\cdot\mathbf{m}\times\dot{\mathbf{m}}\,,\\ 0 &= (\mathbf{h}_{int} + h\,\mathbf{e} - \,\mu\,\dot{\mathbf{m}})\cdot\dot{\mathbf{m}}\,. \end{aligned} \tag{35}$$

Thus, for a Walker process (a solution of (33)) to be a Gilbert process (a solution of (34)), it has to satisfy

$$\begin{aligned} -\,\gamma^{-1}\dot{\mathbf{m}}\cdot\dot{\mathbf{m}} &= (\alpha\triangle\mathbf{m} - \mathbf{T}\mathbf{m} + h\,\mathbf{e})\cdot\mathbf{m}\times\dot{\mathbf{m}}\,,\\ 0 &= (\alpha\triangle\mathbf{m} - \mathbf{T}\mathbf{m})\cdot\dot{\mathbf{m}}\,. \end{aligned} \tag{36}$$

In the next section we show that one such solution exists.

[3] Whenever $c \neq 0$, the physical effect of the part $-c\,\mathbf{c}_1$ of the stray field can be cancelled by the addition of a suitable external field.

[4] An interesting consequence of (33) is that

$$h(\mathbf{m}\cdot\mathbf{e})^{\cdot} \geq 0$$

along all Walker processes.

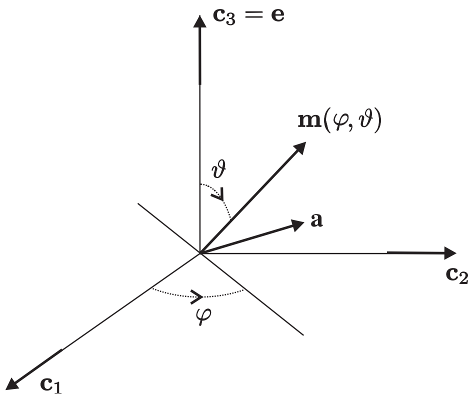

Fig. 2.

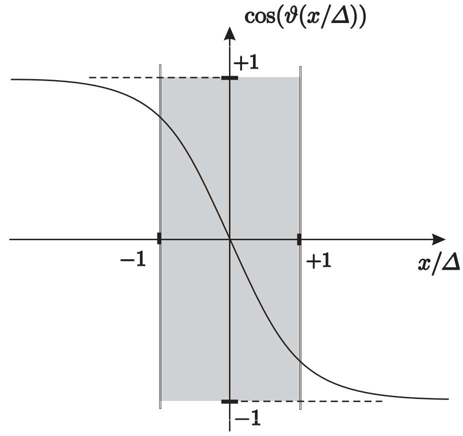

Fig. 3.

4 The Walker solution

4.1 Wall geometry

Let $\mathbf{e}$, φ and ϑ be, respectively, the polar axis and the parallel and meridional coordinates in a system of spherical coordinates. Moreover, let

$$\mathbf{a} = \mathbf{a}(\varphi) = -\sin\varphi\,\mathbf{c}_1 + \cos\varphi\,\mathbf{c}_2 \tag{37}$$

be the unit vector orthogonal to both $\mathbf{e}$ and $\mathbf{m} = \mathbf{m}(\varphi, \vartheta, t)$ and such that $\mathbf{a}\cdot\mathbf{e}\times\mathbf{m} > 0$ (Fig. 2). Then, with the use of the orthonormal basis $(\mathbf{a}, \mathbf{e}, \mathbf{Ae})$, where $\mathbf{A} = \mathbf{A}(\varphi)$ is the skew tensor uniquely associated to $\mathbf{a}$:

$$\mathbf{Ae} = \mathbf{a}\times\mathbf{e} = \cos\varphi\,\mathbf{c}_1 + \sin\varphi\,\mathbf{c}_2\,, \tag{38}$$

we have that

$$\mathbf{m} = \cos\vartheta\,\mathbf{e} + \sin\vartheta\,\mathbf{Ae}\,, \tag{39}$$

$$\dot{\mathbf{m}} = \sin\vartheta\,\dot\varphi\,\mathbf{a} + \dot\vartheta\,\mathbf{Am}\,, \tag{40}$$

where

$$\mathbf{Am} = \mathbf{a}\times\mathbf{m} = -\sin\theta\,\mathbf{e} + \cos\theta\,\mathbf{Ae}\,; \tag{41}$$

hence,

$$\mathbf{m}\times\dot{\mathbf{m}} = \dot\vartheta\,\mathbf{a} - \sin\vartheta\,\dot\varphi\,\mathbf{Am}\,, \tag{42}$$

so that, in particular,

$$\mathbf{e}\cdot\mathbf{m}\times\dot{\mathbf{m}} = \sin^2\vartheta\,\dot\varphi\,. \tag{43}$$

4.2 Satisfying the Walker condition

With (40), the Walker equation (33) becomes:

$$h\,\sin\vartheta\,\dot\vartheta + \mu\,(\sin^2\vartheta\,\dot\varphi^2 + \dot\vartheta^2) = 0\,. \tag{44}$$

We here restrict attention to traveling-wave solutions to (44) being of type (1) and such that

$$\varphi(\xi) = \varphi_o = \text{const.}, \tag{45}$$

so that, in particular,

$$\dot\vartheta = -\,v\,\vartheta'\,.^5 \tag{46}$$

Under the provisional assumptions that both the propagation velocity v and ϑ' be not null and that the signs of v and the datum h be the same, we write equation (44) in the simple form

$$\vartheta'(\xi) = c\,\sin\vartheta(\xi), \quad \xi\in(-\infty,+\infty), \quad c = \frac{h}{\mu v}\cdot \tag{47}$$

Equation (47) is directly reminiscent of the equation derived by Landau and Lifshitz in their classical paper:

$$\vartheta'^2 = \frac{\beta}{\alpha}\,\sin^2\vartheta\,, \tag{48}$$

that is, equation (8) of [1]. The solution of (47) can be read off equation (9) of [1], and is

$$\vartheta(\xi) = \arccos\frac{1-\,\exp(2\,c\,\xi)}{1+\exp(2\,c\,\xi)} \tag{49}$$

(see Fig. 3).

Remark. For a flat domain wall parallel to the easy axis $\mathbf{e}$ and perpendicular to $\mathbf{c}_1$, centered at $\xi = 0$, and of thickness $2\xi_0$, we expect the conditions

$$\mathbf{m}(\mp\xi_0) = \pm\,\mathbf{e}\,. \tag{50}$$

to be satisfied at the boundary. However, Dirichlet-type conditions such as (50) do not seem physically realizable in micromagnetics: instead, Neumann conditions, such as

$$\partial_{\mathbf{c}_1}\mathbf{m}(\mp\xi_0) = \mathbf{0}\,, \tag{51}$$

have a physical sense that is not questioned[6]. Walker's solution effectively concentrates about $\xi = 0$ most of the rotation of $\mathbf{m}$ from $\mathbf{e}$ to $-\,\mathbf{e}$, although it spreads that rotation over the whole real line. Moreover, for any

[5] Here and henceforth a superscript prime denotes differentiation with respect to ξ.

[6] In the present case, the boundary normal is $\mathbf{n} = \pm\,\mathbf{c}_1$ for $\xi = \pm\,\xi_0$; $\partial_{\mathbf{c}_1}\mathbf{m} = (\nabla\mathbf{m})\mathbf{c}_1$.

traveling-wave process of the type we are considering,

$$\nabla \mathbf{m} = \mathbf{m}' \otimes \mathbf{c}_1, \quad \mathbf{m}' = \vartheta' \mathbf{A}\mathbf{m}. \tag{52}$$

Thus, the boundary condition (51) takes for Walker's solution the limit form

$$\vartheta'(\pm\infty) = 0, \tag{53}$$

or rather, with the use of $(47)_1$,

$$\sin\vartheta(\pm\infty) = 0; \tag{54}$$

this last condition, with (39), yields

$$\mathbf{m}(\vartheta(\mp\infty)) = \pm\,\mathbf{e}, \tag{55}$$

in agreement with (50). ♢

4.3 Solving the Gilbert equation

Consider now the vectorial equation (34) and, with (40–43), replace it by the following system of two scalar evolution equations:

$$\begin{aligned} -\gamma^{-1}\dot{\vartheta} &= (\mathbf{h}_{int} + \mathbf{d})\cdot\mathbf{a}, \\ \gamma^{-1}\sin\vartheta\,\dot{\varphi} &= (\mathbf{h}_{int} + \mathbf{d})\cdot\mathbf{A}\mathbf{m} - h\sin\vartheta, \end{aligned} \tag{56}$$

where of course $\mathbf{h}_{int}$ and $\mathbf{d}$ are given by (31) and (15), respectively. This system is equivalent to system (35) and, as we proceed to show, more convenient to arrive to a quick and complete derivation of the traveling-wave solutions to equation (34); the derivations one finds in the literature move instead from (35).

For a steadily propagating magnetization process consistent with (45), that is to say, for $\mathbf{m}(\vartheta(x - vt))$, we have that

$$\dot{\mathbf{m}} = -v\,\mathbf{m}', \quad \triangle\mathbf{m} = \mathbf{m}'', \quad \mathbf{m}'' = \vartheta''\,\mathbf{A}\mathbf{m} - \vartheta'^2\,\mathbf{m}, \tag{57}$$

so that

$$\mathbf{d}\cdot\mathbf{a} = 0, \quad \mathbf{d}\cdot\mathbf{A}\mathbf{m} = \mu v\,\vartheta'; \tag{58}$$

moreover,

$$\begin{aligned} \mathbf{T}\mathbf{m}\cdot\mathbf{a} &= -\sin\varphi_o\cos\varphi_o\sin\vartheta, \\ \mathbf{T}\mathbf{m}\cdot\mathbf{A}\mathbf{m} &= (\beta + \cos^2\varphi_o)\sin\vartheta\cos\vartheta. \end{aligned} \tag{59}$$

Thus, the system (56) reduces to

$$\begin{aligned} v\,\vartheta' &= \gamma\sin\varphi_o\cos\varphi_o\sin\vartheta, \\ 0 &= \alpha\,\vartheta'' - (\beta + \cos^2\varphi_o)\sin\vartheta\cos\vartheta + \mu v\,\vartheta' - h\sin\vartheta. \end{aligned} \tag{60}$$

For the Walker solution (49) of equation (47) to be a solution of this system as well, the second equation must further reduce to

$$\alpha\,\vartheta'' = (\beta + \cos^2\varphi_o)\sin\vartheta\cos\vartheta.^{7} \tag{61}$$

In addition, the so-far indeterminate constants v and φ_o must satisfy the two consistency conditions with the datum h resulting from substitution of (49) and its derivative into, respectively, $(60)_1$ and (61). These conditions are:

$$h = \mu\gamma\sin\varphi_o\cos\varphi_o, \tag{62}$$

and

$$\left(\frac{h}{\mu v}\right)^2 = \frac{\beta + \cos^2\varphi_o}{\alpha}. \tag{63}$$

Just as the Walker condition (47), the first equation of the Gilbert system (60) requires that ϑ' and $\sin\vartheta$ be proportional; for it to be consistent with the second, both (62) and (63) must hold. There is no need to determine the actual shape (49) of the Walker solution to deduce directly from (62)-(63) that, for whatever external field satisfying

$$h < \frac{1}{2}\mu\gamma, \tag{64}$$

the steady propagation of a plane magnetization wave $\mathbf{m}(\vartheta(x - vt))$ is possible, with one or another of the two velocities:

$$v = \frac{h}{\mu}\sqrt{\frac{\alpha}{\beta + \cos^2\varphi_o}}, \tag{65}$$

$$\cos^2\varphi_o = \frac{1}{2}\left(1 \pm \sqrt{1 - \left(\frac{2h}{\mu\gamma}\right)^2}\,\right). \tag{66}$$

Combination of the first of these relations with the last of (47) yields for the constant c the value Δ^{-1}, with

$$\Delta := \sqrt{\frac{\alpha}{\beta + \cos^2\varphi_o}}; \tag{67}$$

we can take Δ, the material parameter that would drive a conceivable sharp-interface asymptotics, as a measure of the *wall thickness*.

Remark. For h, and hence v, equal to zero, (66) gives $\varphi_o = \pi/2$: the wall is a Bloch wall (no stray field), of thickness $\Delta = (\alpha/\beta)^{1/2}$, as Landau and Lifshitz [1] found by solving the extremum problem

$$\int_{-\infty}^{+\infty} \left(\frac{1}{2}\alpha\vartheta'^2 - \frac{1}{2}\beta\cos^2\vartheta\right)\mathrm{d}x = \text{min.}, \tag{68}$$

whose Euler-Lagrange equation is

$$\alpha\,\vartheta'' - \beta\sin\vartheta\cos\vartheta = 0 \tag{69}$$

(*cf.* (61)). ♢

5 High-order exchange energy and exchange dissipation

We now investigate whether the method we propose to generate the Walker solution continues to work for ferromagnets of more general constitutive response than the

[7] Note that the system (36) reduces precisely to the system of this equation and the first of (60).

standard response. We take the expressions for the internal energy density and for the dissipation potential to be

$$\psi_{int} = \frac{1}{2}\,\alpha\,|\nabla \mathbf{m}|^2 + \frac{1}{2}\,\mathbf{Tm}\cdot\mathbf{m} + \frac{1}{2}\,\lambda\,|\triangle \mathbf{m}|^2\,, \quad \lambda > 0\,, \tag{70}$$

and

$$\chi = \frac{1}{2}\,\mu\,|\dot{\mathbf{m}}|^2 + \frac{1}{2}\,\tau\,|\nabla\dot{\mathbf{m}}|^2\,, \quad \tau > 0 \tag{71}$$

(*cf.*, respectively, (28) and (14)). Then, the Walker equation becomes

$$(-\,h\,\mathbf{e} + \mu\,\dot{\mathbf{m}})\cdot\dot{\mathbf{m}} + \tau\,\nabla\dot{\mathbf{m}}\cdot\nabla\dot{\mathbf{m}} = 0\,. \tag{72}$$

Moreover, the internal magnetic field and the dissipation field become

$$\mathbf{h}_{int} = \alpha\,\triangle\mathbf{m} - \mathbf{Tm} - \lambda\,\triangle\triangle\mathbf{m} \tag{73}$$

and

$$\mathbf{d} = -\,\mu\,\dot{\mathbf{m}} + \tau\,\triangle\dot{\mathbf{m}}\,,^{8} \tag{74}$$

so that the corresponding generalized Gilbert equation is

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m}\times(\alpha\triangle\mathbf{m} - \mathbf{Tm} - \lambda\triangle\triangle\mathbf{m} + h\,\mathbf{e} - \mu\,\dot{\mathbf{m}} + \tau\triangle\dot{\mathbf{m}})\,. \tag{75}$$

Remark. The scalar system equivalent to (75) can still be written in the form (56), of course with $\mathbf{h}_{int}$ and $\mathbf{d}$ now given by (73) and (74). The mathematical effects of the high-order exchange terms in (75) have been studied in [8] and [9] (see also [5,6] and [10]). From the physical point of view, the relative importance of the these terms is measured by two additional material parameters, both having the dimensions of a length: these are $\Delta_e := (\lambda/\alpha)^{1/2}$ and $\Delta_d := (\tau/\mu)^{1/2}$. ♢

Under the present circumstances, it is convenient to supplement (52) and (57) with the additional relations

$$\nabla\dot{\mathbf{m}} = -\,v\,\mathbf{m}''\otimes\mathbf{c}_1\,, \quad \triangle\dot{\mathbf{m}} = -\,v\,\mathbf{m}''', \quad \triangle\triangle\mathbf{m} = \mathbf{m}''''\,,$$
$$\mathbf{m}''' = (\vartheta''' - \vartheta'^3)\,\mathbf{Am} - 3\vartheta'\vartheta''\,\mathbf{m}\,, \tag{76}$$
$$\mathbf{m}'''' = (\vartheta'''' - 6\,\vartheta'^2\vartheta'')\mathbf{Am} - (4\vartheta'\vartheta''' - \vartheta'^4 + 3\,\vartheta''^2)\mathbf{m}\,.$$

With the help of these formulae, equation (72) can be written in the following form, when restricted to processes of the type $\mathbf{m}(\vartheta(x - vt))$:

$$-\,h\sin\vartheta\,\vartheta' + \mu v\,\vartheta'^2 + \tau v\,(\vartheta''^2 + \vartheta'^4) = 0\,. \tag{77}$$

It is not difficult to check that this equation admits solutions of the form (47). In fact, the admissible value of the constant $C := c^2$ is the only real and positive solution $C_0 = C_0(\mu, \tau, v, h)$ of the following algebraic system:

$$\frac{\tau}{\mu}\,C^3 + C - \frac{h}{\mu v} = 0. \tag{78}$$

(note that (63) is recovered from (78) for τ equal to zero). It remains for us to check whether the solution we found for (77) also solves the generalized Gilbert equation (75). It is easy to predict a negative outcome. In fact, with the generalized energy density (70), the Landau-Lifshitz functional (68) becomes

$$\int_{-\infty}^{+\infty}(\frac{1}{2}\alpha\vartheta'^2 - \frac{1}{2}\beta\cos^2\vartheta + \frac{1}{2}\lambda\vartheta''^2)\,\mathrm{d}x = \min.\,, \tag{79}$$

and the associated Euler-Lagrange equation,

$$\alpha\,\vartheta'' - \beta\sin\vartheta\cos\vartheta - \lambda\,\vartheta'''' = 0\,, \tag{80}$$

has no solution of type (49). However, to perform a thorough, conclusive check, we observe that, when $\mathbf{m} = \mathbf{m}(\vartheta(x - vt))$, the generalized Gilbert equation reads

$$-v\gamma^{-1}\mathbf{m}' = \mathbf{m}\times(\alpha\,\mathbf{m}'' - \mathbf{Tm} - \lambda\,\mathbf{m}'''' + h\,\mathbf{e} + \mu v\,\mathbf{m}' - \tau v\,\mathbf{m}''')\,, \tag{81}$$

and is equivalent to a system consisting of the first equation of (60) (because both higher-order terms have a null orthogonal projection in the direction of **a**) and the following modification of the second

$$0 = \alpha\,\vartheta'' - (\beta + \cos^2\varphi_o)\sin\vartheta\cos\vartheta + \mu v\,\vartheta' - h\sin\vartheta - \lambda(\vartheta'''' - 6\,\vartheta'^2\vartheta'') - \tau v(\vartheta''' - \vartheta'^3)\,. \tag{82}$$

For these equations to be compatible, once again there must be such a constant C that

$$\vartheta' = C\sin\vartheta\,, \tag{83}$$

with

$$v\,C = \gamma\sin\varphi_o\cos\varphi_o\,, \tag{84}$$

and with the following algebraic condition satisfied whatever the angle ϑ in $(0, \pi)$:

$$0 = (\alpha\,C^2 - (\beta + \cos^2\varphi_o))\cos\vartheta + \mu v\,C - h_{ext} - \lambda\,C^4(12\cos^2\vartheta - 11)\cos\vartheta - \tau v\,C^3(3\cos^2\vartheta - 2), \tag{85}$$

However, this last requirement is impossible to satisfy exactly, unless of course both λ and τ are equal to zero[9].

[8] Note that, in the place of relation (16), we now have

$$\mathbf{d}\cdot\dot{\mathbf{m}} + d = \tau(\triangle\dot{\mathbf{m}} + \nabla\dot{\mathbf{m}}\cdot\nabla\dot{\mathbf{m}}) = \tau\,\mathrm{div}\,((\nabla\dot{\mathbf{m}})^T\dot{\mathbf{m}})\,.$$

[9] Relation (83) has the following differential consequences:

$$\begin{aligned}\vartheta'' &= C^2\sin\vartheta\cos\vartheta,\\ \vartheta''' &= C^3(\cos^2\vartheta - \sin^2\vartheta)\sin\vartheta,\\ \vartheta'''' &= C^4(\cos^2\vartheta - 5\,\sin^2\vartheta)\sin\vartheta\cos\vartheta,\end{aligned} \tag{86}$$

whence

$$\begin{aligned}\vartheta'''' - 6\vartheta'^2\vartheta'' &= C^4(\cos^2\vartheta - 11\sin^2\vartheta)\sin\vartheta\cos\vartheta\\ \vartheta''' - \vartheta'^3 &= C^3(\cos^2\vartheta - 2\sin^2\vartheta)\sin\vartheta\,.\end{aligned} \tag{87}$$

Condition (85) obtains when we substitute (87) into (82).

6 Dry-friction dissipation

We now propose a generalization of Walker's solution to the case when a *dry-friction* term is included in the dissipation vector[10]. Precisely, we take the dissipation potential to be

$$\chi = \frac{1}{2}\,\mu\,|\dot{\mathbf{m}}|^2 + \eta\,|\dot{\mathbf{m}}|\,, \tag{88}$$

with μ and η positive constants, so that the dissipation vector $\mathbf{d}$ becomes

$$\mathbf{d} = -\,\mu\,\dot{\mathbf{m}} + \eta\,\mathbf{f}(\dot{\mathbf{m}}) \tag{89}$$

where

$$\begin{aligned} &-\mathbf{f}(\dot{\mathbf{m}}) = |\dot{\mathbf{m}}|^{-1}\dot{\mathbf{m}} \quad \text{for } \dot{\mathbf{m}} \neq \mathbf{0}\,, \\ &-\mathbf{f}(\mathbf{0}) \in \{\mathbf{v}\,|\,|\mathbf{v}| \leq 1\}\,, \end{aligned} \tag{90}$$

is the *dry-friction mapping*. Next, we replace the prescription (32) for the applied magnetic field by the following more general prescription:

$$\mathbf{h}_{ext} = h\,\mathbf{e} + h_a\mathbf{a} + h_{Ae}\mathbf{A}\mathbf{e}, \tag{91}$$

where all three components of $\mathbf{h}_{ext}$, not only h, are control parameters we can assign the constant values we wish. With (89–91), the Walker condition (11) takes the form

$$\begin{aligned} &-h_a \sin\vartheta\,\dot{\varphi} + (h\sin\theta - h_{Ae}\cos\theta)\dot{\vartheta} \\ &+\mu(\sin^2\vartheta\,\dot{\varphi}^2 + \dot{\vartheta}^2) + \eta\,|(\sin^2\vartheta\,\dot{\varphi}^2 + \dot{\vartheta}^2)^{1/2}| = 0\,, \end{aligned} \tag{92}$$

and the simpler form

$$-\,h\sin\theta + h_{Ae}\cos\theta + \mu v\,\vartheta' + \eta\,\mathrm{sign}\,(v\vartheta') = 0 \tag{93}$$

for processes of the type $\mathbf{m}(\vartheta(x-vt))$. Likewise, with (91) we can write the Gilbert system as

$$\begin{aligned} -\gamma^{-1}\,\dot{\vartheta} &= (\mathbf{h}_{int} + \mathbf{d})\cdot\mathbf{a} + h_a\,, \\ \gamma^{-1}\sin\vartheta\,\dot{\varphi} &= (\mathbf{h}_{int} + \mathbf{d})\cdot\mathbf{A}\mathbf{m} - h\sin\vartheta + h_{Ae}\cos\vartheta\,; \end{aligned} \tag{94}$$

with (89) and (90) and for processes of the type $\mathbf{m}(\vartheta(x-vt))$, this system becomes

$$\begin{aligned} v\,\vartheta' &= \gamma\,\sin\varphi_o\cos\varphi_o\sin\vartheta - \gamma h_a\,, \\ 0 &= \alpha\,\vartheta'' - (\beta + \cos^2\varphi_o)\sin\vartheta\cos\vartheta + \mu v\,\vartheta' - h\sin\vartheta \\ &\quad + h_{Ae}\cos\vartheta + \eta\,\mathrm{sign}\,(v\vartheta')\,. \end{aligned} \tag{95}$$

As a glance to (93) and the first of (95) makes evident, no solution of the former equation can also solve the latter, unless perhaps $h_{Ae} = 0$. Now, mutual consistency of the equations (95) is guaranteed provided that the constants h, v and φ_o satisfy the conditions (62–63) and that, in addition, the remaining components of the control field be such that

$$h_a = \frac{\eta}{\mu\gamma}\mathrm{sign}\,(v\vartheta')\,, \quad h_{Ae} = \frac{\alpha\eta}{(\mu v)^2}\,h\,\mathrm{sign}\,(v\vartheta')\,. \tag{96}$$

Thus, for h_{Ae} to be null, h should be null as well, a circumstance when, as is easily seen, the system (95) has no solution for $\eta \neq 0$. We must then conclude that, in the presence of dry friction, no Walker process solves the Gilbert equation. This notwithstanding, an explicit solution to the Gilbert equation can be found. Assuming that all consistency conditions hold, we can write $(95)_1$ in the form

$$\vartheta'(\xi) = \frac{1}{\Delta}(\sin\vartheta(\xi) + r)\,, \quad r = -\frac{\eta}{|h|}\,\mathrm{sign}\vartheta'\,, \tag{97}$$

with Δ given by (67).

An easy continuity argument shows that the sign of ϑ' must be constant for class-C^1 solutions of (97)[11]. Hence, we treat r in (97) as a constant parameter. Granted this, solutions of (97) exist only for $|r| < 1$, and have the form

$$\vartheta(\xi) = -\mathrm{sign}\,(r)\arccos\frac{1 - f^2(\xi)}{1 + f^2(\xi)}, \tag{98}$$

(*cf.* (49)), where

$$f(\xi) = \frac{f_2\,F\exp{(\xi/\Delta_r)} - f_1}{F\exp{(x/\Delta_r)} - 1}\,, \tag{99}$$

$$f_1 = \frac{-1 + \sqrt{1 - r^2}}{r}\,, \quad f_2 = \frac{-1 - \sqrt{1 - r^2}}{r}\,, \tag{100}$$

$$F = \frac{1 - |r| - \sqrt{1 - r^2}}{1 - |r| + \sqrt{1 - r^2}}\,; \tag{101}$$

moreover,

$$\Delta_r = \frac{1}{\sqrt{1 - r^2}}\,\Delta \tag{102}$$

is the thickness of the transition layer when dry-friction is accounted for.

Note that, for r small, $f_1 \approx -r/2$, $f_2 \approx -2/r$ and $F \approx r/2$; therefore, when the dry-friction coefficient η is small, $f(\xi) \approx -\exp(\xi/\Delta)$ and Walker's solution is recovered. Note also that the wall thickness becomes larger when the dry-friction coefficient increases (Fig. 4). Finally, as to the limit values of the magnetization, one finds that

$$\lim_{\xi\to\pm\infty}\mathbf{m}(\xi) = \pm\sqrt{1 - r^2}\,\mathbf{e} + r\,\mathbf{e}\times\mathbf{a}\,. \tag{103}$$

We thank Vanda Valente for useful discussions, and Alexei Bogdanov for pointing out to us the reference [6]. We also thank the reviewers for valuable comments that prompted us to improve our exposition. This work has been supported by Progetto Cofinanziato 2000 "Modelli Matematici per la Scienza dei Materiali" and by TMR Contract FMRX-CT98-0229 "Phase Transitions in Crystalline Solids".

[10] This term should accommodate possible "slip-stick" motions of domain walls, whence the name.

[11] Interestingly, it also follows from (97) that, for class-C^0 solutions, the jump in ϑ' must equal $2\,|r|$.

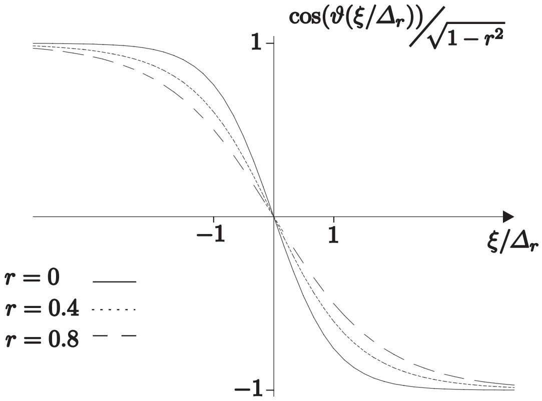

Fig. 4.

Appendix

1. Notation and units

In SI units, the Gilbert equation reads

$$\gamma_e^{-1}\dot{\mathbf{m}} = \mu_0 \mathbf{m} \times \mathbf{H}_{eff}\,, \tag{104}$$

where $\gamma_e = g\,q_e/m_e$ is the *gyromagnetic ratio*, $q_e < 0$ is the *signed electron charge*, m_e is the *electron mass* and $g > 0$ is the *Landé factor*; μ_0 denotes the magnetic permeability of the vacuum, and

$$\mathbf{H}_{eff} = -(\mu_0 M_s)^{-1}\delta_{\mathbf{m}}\Gamma\,, \tag{105}$$

is the *effective field*, with M_s the saturation magnetization and Γ the *Gibbs free energy*. The standard free energy is written as

$$\Gamma = \int_\Omega A|\nabla\mathbf{m}|^2 - K(\mathbf{e}\cdot\mathbf{m})^2 - \mu_0 M_s\left(\frac{1}{2}\,\mathbf{H}_s + \mathbf{H}_{ext}\right)\cdot\mathbf{m}\,, \tag{106}$$

where $\mathbf{H}_s$ and $\mathbf{H}_{ext}$ are, respectively, the stray field and the external magnetic field, A is the exchange constant and K is the anisotropy constant.

Our notation is recovered upon defining the dimensionless quantities

$$\begin{aligned}\psi &= (\mu_0 M_s^2)^{-1} g\,,\\ \beta &= 2(\mu_0 M_s^2)^{-1} K\,,\\ \mathbf{h} &= M_s^{-1}\mathbf{H}_{eff}\,,\\ \mathbf{h}_s &= M_s^{-1}\mathbf{H}_s\,,\\ \mathbf{h}_{ext} &= M_s^{-1}\mathbf{H}_{ext}\,,\end{aligned} \tag{107}$$

and

$$\begin{aligned}\alpha &= 2(\mu_0 M_s^2)^{-1} A\,,\\ \gamma &= M_s\mu_0\gamma_e\end{aligned} \tag{108}$$

which have, respectively, the dimensions of (length)^2 and (time)$^{-1}$.

2. The stray field in a flat wall

To prove (26), we introduce the usual representation of the stray field in terms of the scalar potential H:

$$\mathbf{h}_s = -\nabla H\,; \tag{109}$$

we note that, due to (25),

$$\operatorname{div}\mathbf{m} = \mathbf{m}'\cdot\mathbf{c}_1 = (\mathbf{m}\cdot\mathbf{c}_1)'\,; \tag{110}$$

and, finally, we write equation $(23)_2$ as

$$\triangle H = (\mathbf{m}\cdot\mathbf{c}_1)', \tag{111}$$

where, for t fixed, the right side depends at most on x. But then the representation formula for the solutions of the Poisson equation implies that $H = H(x,t)$, and hence the stray field is parallel to $\mathbf{c}_1$:

$$\mathbf{h}_s = -H'\,\mathbf{c}_1\,, \quad H'(x,t) = \mathbf{m}(x,t)\cdot\mathbf{c}_1 + c(t)\,. \tag{112}$$

References

1. L.D. Landau, E. Lifshitz, Phys. Z. Sowjet. **8**, 153 (1935). Reprinted in *Collected Papers of L.D. Landau*, edited by D. ter Haar (Pergamon Press, 1965), pp. 101–114.
2. L.R. Walker, unpublished; *cf.* J.F. Dillon, Jr. in *A Treatise on Magnetism,* Vol. III, edited by G.T. Rado, H. Suhl (Academic Press, 1963) pp. 450–453.
3. A. Hubert, R. Schäfer, *Magnetic Domains* (Springer, 1998).
4. T.L. Gilbert, Phys. Rev. **100**, 1243 (1955).
5. P. Podio-Guidugli, Eur. Phys. J. B **19**, 417 (2001).
6. A. Hubert, *Theorie der Domänenwände in Geordneten Medien* (Springer, 1974) *(Theory of Domain Walls in Ordered Media).*
7. A. DeSimone, P. Podio-Guidugli, Arch. Rational Mech. Anal. **136**, 201 (1996).
8. M. Bertsch, P. Podio-Guidugli, V. Valente, Annali Mat. Pura Appl. (IV) **CLXXIX**, 331 (2001).
9. P. Podio-Guidugli, V. Valente, Nonlinear Analysis **47**, 147 (2001).
10. P. Podio-Guidugli, V. Valente, in preparation.
11. A. Visintin, Physica B **233**, 365 (1997).